Integral transforms in the two-dimensional non-linear formulation of longitudinal fins with variable profile

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Received August 1994
Revised July 1996

Nomenclature

A_{ij} = matrix coefficients defined by equation (9b)
B_{ij} = matrix coefficients defined by equation (9c)
Bi = Biot number
b = coefficient of the variable thermal conductivity, dimensionless
C = aspect ratio
fi = transformed boundary condition
h = heat transfer coefficient
k = thermal conductivity
L = fin length
N = truncation order in eigenfunction expansions
N_i = normalization integral
Q_0 = dimensionless heat transfer rate
r = dimensionless thickness of the fin tip
$T(x,y)$ = temperature distribution
T_b = fin base temperature
$T\infty$ = surroundings temperature
x,X = dimensional and dimensionless longitudinal co-ordinate
y,Y = dimensional and dimensionless transversal co-ordinate

Greek symbols

α = local inclination of variable profile
β = coefficient of the variable thermal conductivity, equation (3a)
$\in$ = relative error in the two-dimensional solution
$\in_0$ = thickness at the fin base
$\in_f$ = thickness at the fin tip
$\in(X)$ = dimensionless variable profile
ψ = eigenfunctions of problem (3)
μ_i = eigenvalues of problem (3)
$\theta(X,Y)$ = dimensionless temperature distribution

Subscripts and superscripts

av = average temperature
i,j = order of related eigenquantities
– = transformed quantities

Introduction

Extended surfaces (or fins) play a major role in the design of heat exchange devices within different fields of application, aimed at providing a heat transfer enhancement effect through an increase in the total exchange area. Although fins of uniform longitudinal profile are more commonly employed, owing to the obvious advantages in manufacturing and installation, extended surfaces of variable profiles are also of great practical interest, in connection with optimized designs towards the minimization of the fin weight and consequent reduction on the fin material utilization (Kern and Kraus, 1972; Snider and Kraus, 1983).

International Journal for Numerical Methods for Heat & Fluid Flow
Vol. 8 No. 1, 1998, pp. 27–42.

The classical one-dimensional fin approximation has been extensively recalled in the literature and explicit solutions are available for uniform longitudinal profiles and a limited class of variable profiles (Kern and Kraus, 1972; Mikhailov and Ozisik, 1984; Snider and Kraus, 1983) which significantly simplify the thermal analysis of finned devices.

A few contributions are available on the error estimation for these simplified solutions in the case of fins with constant profile (Irey, 1968; Lau and Tan, 1973), including a recently proposed modified one-dimensional formulation (Aparecido and Cotta, 1990a), which reduces the error by an order of magnitude in comparison with the classical fin formulation. However, for the more general situation of a longitudinally variable profile, applicability limits for the few simplified formulations are not clearly established, in part due to difficulties in the accurate solution of the two dimensional heat conduction problem within an irregular domain, not in general achievable through analytical methods (Mikhailov and Ozisik, 1984), especially if the effect of variable thermal conductivity is accounted for.

The present work brings a general hybrid numerical-analytical solution for longitudinal fins with arbitrarily variable profile and temperature dependent thermal conductivity, by handling the associated steady-state two-dimensional heat conduction problem, and providing numerical results with full accuracy control. The generalized integral transform technique (GITT) (Cotta, 1992) is the basic tool behind such development, which has been gradually advanced towards the error-controlled solution of different classes of problems in heat and fluid flow, including various non-linear diffusion and convection-diffusion problems (Cotta, 1992). The problem considered here is an extension to previous developments on diffusion and convection within irregular geometries (Aparecido and Cotta, 1990b, 1990c, 1992; Aparecido *et al.,* 1989), including the non-linear nature introduced through the variable thermal conductivity.

The basic steps in application of the integral transform method are the choice of an appropriate auxiliary eigenvalue problem, which provides the basis for the eigenfunction expansion, and subsequent integral transformation of the original partial differential problem. An infinite system of ordinary differential equations then results, which is truncated to an automatically controlled finite order, for computational purposes, and numerically handled through boundary value problem solvers with global error control and estimation schemes, readily available in scientific subroutines libraries (IMSL Library, 1987). The explicit inversion formula is then recalled to provide an analytic representation of the original potential anywhere within the domain. This approach is illustrated here for some typical variable profiles, and the excellent convergence behaviour of the proposed expansions is demonstrated. The influence of the governing parameters, Biot number and aspect ratio is also investigated and the errors involved in the classical one-dimensional approximation are more clearly identified.

It is also demonstrated that the integral transform solution of the related two-dimensional problem can provide reliable approximate solutions, by

retaining just the first term in the expansions for the average temperature. The resulting expressions are as simple as those obtained through the classical one-dimensional formulation, based on the solution of a single second order ordinary differential equation, and the numerical results show a more consistent behaviour, especially for increasing values of Biot number, aspect ratio and degree of non-linearity

Analysis

We consider the two-dimensional heat conduction equation, for steady-state and temperature-dependent thermal conductivity, written for a longitudinal fin of variable profile, according to Figure 1. The temperature at the fin base is assumed uniform and heat losses through the fin tip are disregarded. In dimensionless form, the problem formulation is given as:

$$\frac{\partial}{\partial Y}\left[K(\theta)\frac{\partial\theta(X,Y)}{\partial Y}\right]+\frac{\partial}{\partial X}\left[K(\theta)\frac{\partial\theta(X,Y)}{\partial X}\right]=0\,,\quad 0<X<C\,,\,0<Y<\epsilon(X) \tag{1a}$$

with boundary conditions

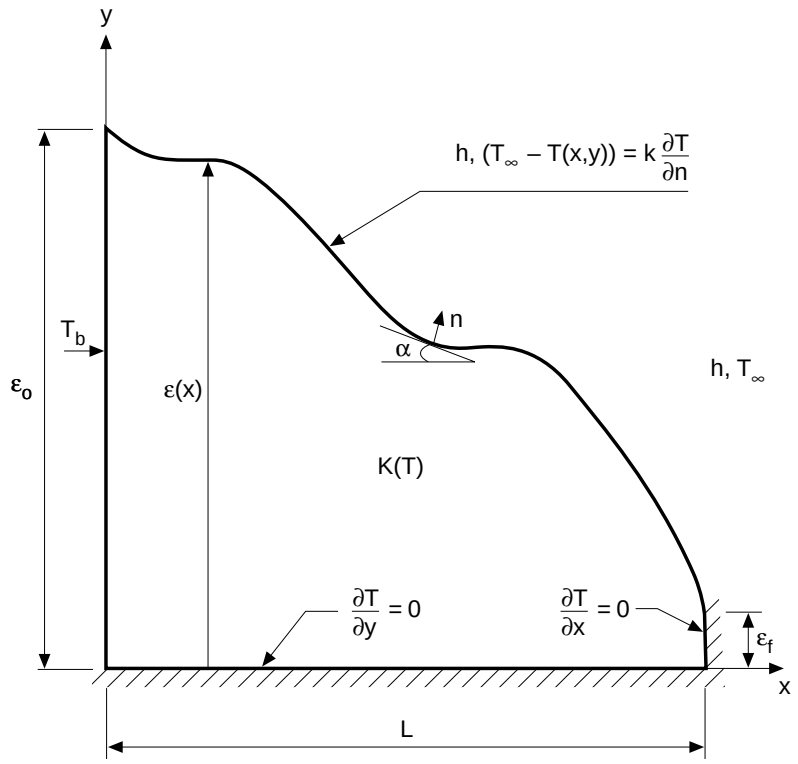

Figure 1. Geometry and co-ordinate system for a longitudinal fin with variable profile

$$\theta(0,Y)=1 \quad ; \quad \left.\frac{\partial\theta(X,Y)}{\partial X}\right|_{X=C}=0 \tag{1b,c}$$

$$\left.\frac{\partial\theta(X,Y)}{\partial Y}\right|_{Y=0}=0 \; ; \left[K(\theta)\left(\sin\alpha\frac{\partial\theta}{\partial X}+\cos\alpha\frac{\partial\theta}{\partial Y}\right)\right]_{Y=\in(X)}+Bi\theta(X,\in(X))=0 \tag{1d,e}$$

where the various dimensionless groups are defined by

$$X=\frac{x}{\in_0} \quad ; \quad Y=\frac{y}{\in_0} \quad ; \quad \theta(X,Y)=\frac{T(x,y)-T_\infty}{T_b-T_\infty}$$

$$C=\frac{L}{\in_0} \quad ; \quad Bi=\frac{h\in_0}{k_\infty} \quad ; \quad k(\theta)=\frac{k(T)}{k_\infty} \tag{2}$$

and α is the local inclination of the variable profile.

If the temperature-dependent thermal conductivity is considered in the usual linear form

$$k(T)=k_\infty\left(1+\beta\left(T-T_\infty\right)\right) \tag{3a}$$

in order to reduce the number of parameters to be studied; then, the dimensionless function $K(\theta)$ becomes

$$K(\theta)=1+b\theta \tag{3b}$$

where,

$$b=\beta(T_b-T_\infty) \tag{3c}$$

The application of Kirchoff's transformation (Mikhailov and Ozisik, 1984), in the non-linear problem (1) above is convenient to the analysis that follows. Therefore, the following dependent variable is defined:

$$U(X,Y)=\int_0^\theta K(\theta')d\theta' \tag{4a}$$

or,

$$U=\theta+\frac{b}{2}\theta^2 \tag{4b}$$

and problem (1) is rewritten as

$$\frac{\partial^2U(X,Y)}{\partial X^2}+\frac{\partial^2U(X,Y)}{\partial Y^2}=0, \quad 0<X<C, \quad 0<Y<\in(X) \tag{5a}$$

with boundary conditions

$$U(0,Y) = 1 + \frac{b}{2}\,; \quad \left.\frac{\partial U(X,Y)}{\partial X}\right|_{X=C} = 0 \tag{5b,c}$$

$$\left.\frac{\partial U(X,Y)}{\partial Y}\right|_{Y=0} = 0 \tag{5d}$$

$$\sin\alpha \left.\frac{\partial U}{\partial X}\right|_{Y=\epsilon(X)} + \cos\alpha \left.\frac{\partial U}{\partial Y}\right|_{Y=\epsilon(X)} + BiU(X,\epsilon(X)) = \frac{Bib}{2}\theta^2(X,\epsilon(X)) \tag{5e}$$

It should be noted that the transformation does not eliminate the non-linear nature of the original problem, represented in the source term of boundary condition (5e), but simplifies the operators in the partial differential equation (5a).

The application of the integral transform method (Cotta, 1992) to the present problem involves a combination of previous developments on the solution of diffusion problems within irregular domains (Aparecido and Cotta, 1990b, 1990c, 1992; Aparecido *et al.*, 1989) and of non-linear diffusion and convection-diffusion problems (Cotta, 1990; Cotta and Serfaty, 1991; Serfaty and Cotta, 1990, 1992). First, an appropriate auxiliary eigenvalue problem is selected, which provides the basis for the eigenfunction expansion. Following the ideas in such previous developments, the eigenvalue problem is taken as:

$$\frac{\partial^2 \psi_i}{\partial Y^2} + \mu_i^2 \psi_i(Y;X) = 0\,, \quad 0 < Y < \epsilon(X) \tag{6a}$$

with boundary conditions

$$\left.\frac{\partial \psi_i(Y;X)}{\partial Y}\right|_{Y=0} = 0\,; \quad \left[\cos\alpha \frac{\partial \psi_i(Y;X)}{\partial Y}\right]_{Y=\epsilon(X)} + Bi\psi_i(\epsilon(X);X) = 0 \tag{6b,c}$$

The longitudinal co-ordinate, X, appears as a parameter in the auxiliary problem proposed for the transversal co-ordinate, Y, owing to the X-dependence in the variable domain of the transversal co-ordinate. It should also be noted that the boundary condition (6c) does not incorporate the X-component of the normal derivative at the boundary $Y = \epsilon(X)$, as well as the non-linear term, which will be recovered after the integral transformation procedure. Problem (6) allows definition of the integral transform pair below

$$\overline{U}_i(X) = \frac{1}{N_i}\int_0^{\epsilon(X)} \psi_i(Y;X)U(X,Y)dY\,, \text{ transform} \tag{7a}$$

$$U(X,Y) = \sum_{i=1}^{\infty} \psi_i(Y;X)\overline{U}_i(X)\,, \text{ inverse} \tag{7b}$$

where the X-dependent normalization integral is computed from

$$N_i(X) = \int_0^{\epsilon(X)} \psi_i(Y;X)dY \tag{7c}$$

The eigenvalue problem (6) is solved in analytic form to yield:

$$\psi_i(Y;X) = \cos(\mu_i(X)Y) \text{ , for the eigenfunctions} \tag{8a}$$

$$\mu_i(X)\tan\left[\mu_i(X)\epsilon(X)\right] = \frac{Bi}{\cos\alpha}\text{, for the eigenvalues} \tag{8b}$$

$$N_i(X) = \frac{\epsilon(X)\left[\mu_i^2 + \left(\frac{Bi}{\cos\alpha}\right)^2\right] + \frac{Bi}{\cos\alpha}}{2\left[\mu_i^2 + \left(\frac{Bi}{\cos\alpha}\right)^2\right]}\text{ , for the norms} \tag{8c}$$

The integral transform approach proceeds by operating on equation (5a) with the operator

$$\frac{1}{N_i(X)}\int_0^{\epsilon(X)} \psi_i(Y;X)dY$$

to provide, after manipulations with the inverse formula (7b) and boundary conditions:

$$\frac{d^2\overline{U}_i(X)}{dX^2} = -\frac{1}{N_i}\sum_{j=1}^{\infty}\left\{\left[2B_{ij} - \tan\alpha\left(\psi_i\psi_j\right)_{Y=\epsilon(X)}\right]\frac{d\overline{U}_j}{dX} + \left[A_{ij} - \tan\alpha\left(\psi_i\frac{\partial\psi_j}{\partial X}\right)_{Y=\epsilon(X)} - \delta_{ij}N_i\mu_i^2\right]\overline{U}_j\right\} -$$

$$-\frac{Bi}{\cos\alpha N_i b}\left\{1 + b\sum_{j=1}^{\infty}\psi_j(\epsilon(X);X)\overline{U}_j - \sqrt{1 + 2b\sum_{j=1}^{\infty}\psi_j(\epsilon(X);X)\overline{U}_j}\right\}\psi_i(\epsilon(X);X);\quad i = 1,2,.... \tag{9a}$$

where,

$$A_{ij}(X) = \int_0^{\epsilon(X)} \psi_i\frac{\partial^2\psi_j}{\partial X^2}dY \tag{9b}$$

$$B_{ij}(X) = \int_0^{\epsilon(X)} \psi_i\frac{\partial\psi_j}{\partial X}dY$$

$$\text{and } \delta_{ij} = \begin{cases} 0, \text{ for } i \neq j \\ 1, \text{ for } i = j \end{cases}. \tag{9c}$$

The required boundary conditions to solve equation (9a) are obtained through the same integral transformation of equations (5b,c), to yield

$$\bar{U}_i(0) = \left(1 + \frac{b}{2}\right)\bar{f}_i, \quad \text{i=1,2,...} \tag{9d}$$

$$\left.\frac{d\bar{U}_i}{dX}\right|_{X=C} + \frac{1}{N_i(C)}\sum_{j=1}^{\infty} B_{ij}(C)\bar{U}_j(C) = 0, \quad \text{i=1,2,...} \tag{9e}$$

where,

$$\bar{f}_i = \frac{\sin\mu_i(0)}{N_i(0)\mu_i(0)} \tag{9f}$$

For computational purposes, the infinite *ODE* system (9a,d,e) is truncated to a sufficiently large finite order N, as discussed in the next section, and numerically handled through well-established boundary value problem solvers with automatic error control (IMSL Library, 1987). Once the transformed potentials, $\bar{U}_i'$s, have been obtained for different positions in the X co-ordinate, the inversion formula (7b) and equation (4b) are recalled to provide an analytical expression for the dimensionless temperature field at any position Y of interest. Also, from this analytical expression, the average temperature at any cross section is readily obtained as:

$$\theta_{av}(X) = \frac{1}{\epsilon(X)}\int_0^{\epsilon(X)} \theta(X,Y)dY \tag{10}$$

Computational procedure

In the present hybrid approach, the numerical task is reduced to the solution of a system of second order ordinary differential equations for the transformed potentials, represented by equations (9). A number of algorithms for boundary value problems are available that implement automatic error control through adaptive mesh refinement schemes. One such algorithm is available in subroutine *DBVPFD* of the IMSL library (1987), which controls the local relative error in order to satisfy the user prescribed accuracy for the numerical results. In addition, for highly non-linear problems, a continuation procedure is available through parametrization of the non-linear terms. Therefore, since the numerical results for the transformed potentials are obtained within prescribed accuracy, one is left with the need to select the truncation order, N automatically, in the eigenfunctions expansions, in order to devise a fully error-controlled algorithm. Following the ideas in the computational procedures advanced in Cotta (1992), for the case of elliptic problems such as the one under consideration, the truncation order N is increased in fixed steps ΔN and a testing scheme is employed at each position (X,Y) of interest, until convergence has been achieved within the user requested accuracy. The simple testing formula is written as:

$$\epsilon = \max_{(X,Y)} \left| \frac{\sum_{i=N+1}^{N+\Delta N} \psi_i(Y;X)\bar{U}_i(X)}{\sum_{i=1}^{N+\Delta N} \psi_i(Y;X)\bar{U}_i(X)} \right| \tag{11}$$

and N is increased until $\epsilon < Tol$ where Tol is the user prescribed accuracy target. Also, for improved computational performance, the numerical results obtained for the lower truncation order N serve as excellent initial guesses for the next run with an increased truncation order ($N + \Delta N$), speeding up considerably the automatic error control scheme.

Results and discussion

Initially, the automatic error control feature of the integral transform approach was reconfirmed through the case of a rectangular profile $\epsilon(X) = const.$ and constant thermal conductivity, which has a straightforward exact analytical solution. As expected, the agreement was perfect to within the requested relative error target (10^{-5}).

Next, the case of a trapezoidal profile with constant thermal conductivity was considered, which is represented by the following equation

$$\epsilon(X) = 1 + \frac{X}{C}(r-1) \tag{12a}$$

where,

$$r = \frac{\epsilon_f}{\epsilon_0} \quad ; \quad \tan\alpha = \frac{1-r}{c} \tag{12b,c}$$

The special case of a triangular fin (Aziz and Nguyen, 1992) is then recovered by letting r→0.

Table I illustrates the convergence behaviour of the proposed eigenfunction expansion for the average temperature, equation (10), in the case of a trapezoidal fin of constant thermal conductivity for different values of Biot number, Bi, and aspect ratio, C. Also shown are the results from the classical one-dimensional formulation for a trapezoidal fin (Chung *et al.*, 1989). It is clearly noticeable that the convergence rates are somehow improved for lower values of Bi and higher values of C, since then the X-component of the boundary normal derivative becomes less significant. As previously mentioned, this component is not incorporated in the eigenvalue problem boundary condition, and appears as a "source term" in the system for the transformed potential, therefore affecting the convergence rates, as usual in eigenfunction expansion-type approaches. However, the overall convergence behaviour is still quite outstanding, at least in the range of the parameters, typical of extended surfaces applications, considered here. When required, convergence acceleration schemes can be recalled (ISML Library, 1987) either through

N	*X* = 0.000	*X* = 0.750	*X* = 1.875	*X* = 3.000
(*C = 3.0; r = 10^{-3}*)				
Bi = 1.00				
1 (% error)	1.000	0.5144 (1.67)	0.1483 (3.97)	0.0185 (9.44)
4	1.000	0.5067	0.1431	0.0171
7	1.000	0.5062	0.1428	0.0170
10	1.000	0.5060	0.1427	0.0169
(*) (% error)	1.000	0.4726 (7.06)	0.1195 (19.4)	0.0134 (24.4)
Bi = 0.10				
1 (% error)	1.0000	0.8465 (0.50)	0.6401 (1.30)	0.4711 (2.53)
4	1.0000	0.8427	0.6357	0.4615
7	1.0000	0.8424	0.6350	0.4606
10	1.0000	0.8423	0.6348	0.4603
(*) (% error)	1.0000	0.8407 (0.19)	0.6316 (0.51)	0.4571 (0.70)
Bi = 0.01				
1 (% error)	1.0000	0.9783 (0.08)	0.9462 (0.19)	0.9151 (0.31)
4	1.0000	0.9776	0.9446	0.9126
7	1.0000	0.9776	0.9444	0.9123
10	1.0000	0.9775	0.9444	0.9122
(*) (% error)	1.0000	0.9775 (0.01)	0.9442 (0.02)	0.9120 (0.03)

Note: (*) One-dimensional solution (Chung *et al.*, 1989)

N	*X* = 0.000	*X* = 3.750	*X* = 6.250	*X* = 10.000
(C = 10; r = 10^{-3})				
Bi = 1.00				
1 (% error)	1.000	0.0267 (0.75)	0.0011 (0.00)	0.0000 (0.00)
4	1.000	0.0264	0.0011	0.0000
7	1.000	0.0265	0.0011	0.0000
10	1.000	0.0265	0.0011	0.0000
(*) (% error)	1.000	0.0169 (–36.62)	0.0005 (–51.01)	0.0000 (–89.94)
Bi = 0.10				
1 (% error)	1.0000	0.3052 (0.25)	0.1143 (0.42)	0.0118 (0.94)
4	1.0000	0.3045	0.1138	0.0117
7	1.0000	0.3044	0.1138	0.0117
10	1.0000	0.3044	0.1138	0.0117
(*) (% error)	1.0000	0.3001 (–1.67)	0.1113 (–2.58)	0.0113 (–3.68)
Bi = 0.01				
1 (% error)	1.0000	0.7588 (0.07)	0.6193 (0.12)	0.4408 (0.22)
4	1.0000	0.7583	0.6186	0.4408
7	1.0000	0.7583	0.6185	0.4399
10	1.0000	0.7583	0.6185	0.4398
(*) (% error)	1.0000	0.7580 (–0.04)	0.6182 (–0.05)	0.4395 (–0.07)

Note: (*) One-dimensional solution (Chung *et al.*, 1989)

Table I.
Convergence of the eigenfunction expansion and comparison against the classical one-dimensional formulation

filtering solutions or integral balance procedures, with additional analytical involvement. The results presented for $N = 10$ in Table I are fully converged to ±1 in the last digit given, achieved through a user prescribed accuracy of 10^{-5} in the *ODE*'s solver. A typical run with $N < 10$ takes about 75 seconds of CPU time on the IBM4381 mainframe computer, in the least favourable situation.

Table I also presents the classical one-dimensional fin formulation (Serfaty and Cotta, 1990) compared with the full two-dimensional formulation obtained here. As expected, the error increases with the Biot number as the temperature gradients in the transversal direction become more pronounced, exceeding the value of 20 per cent for $Bi = 1.0$. The influence of the aspect ratio is less marked but significant, from a comparison of the results for $C = 3.0$ and 10.0 at the same value of $Bi = 0.1$. It is also of interest to observe the behaviour of the two-dimensional solution with one single term in the eigenfunction expansion ($N = 1$). From the mathematical point of view, this solution is as simple as the classical one-dimensional formulation, i.e. it is obtained essentially from the solution of a second order ordinary differential equation. Nevertheless, the error for this alternative approximate solution ($N = 1$) is better behaved than that for the one-dimensional solution. This behaviour is even more noticeable for higher values of Bi (= 1.0) and/or C (= 10.0), indicating that in the realm of applications and for optimization studies of reduced cost, this approximate solution might offer a more reliable alternative to the classical approach.

Such aspects of convergence rates and relative accuracy are also observable in graphical form, through Figures 2 and 3, for the average temperature distribution along the entire length of the fin. Figure 2 shows the convergence behaviour for the case $C = 3.0$ and $Bi = 0.1$, demonstrating the practically coincident results, to the graph scale, for $N > 4$. In addition, the results for one single term in the expansion ($N = 1$) are in very good agreement with the one-dimensional solution over the whole domain. Figure 3, again for $C = 3.0$, brings the fully converged two-dimensional solutions for different Biot numbers ($Bi = 0.01$, 0.1 and 1.0), compared against the classical one-dimensional solution (Chung *et al.*, 1989), and demonstrates the increase of the error in the approximate solution as the value of Biot increases.

Results were also obtained for a longitudinal fin with concave parabolic profile, again for constant thermal conductivity, defined by the following expression:

$$\in (X) = \left(\frac{X}{C} - 1 \right)^2 \tag{13}$$

In Mikhailov and Özisik (1984), the classical one-dimensional formulation was employed in the solution of concave parabolic fins with negligible thickness at the fin tip. In order to allow for critical comparisons with the present two-dimensional results, the computations were performed with $r = 10^{-5}$, which was verified to represent adequately the situation of $r \rightarrow 0$, or negligible heat transfer area at the tip. Numerical results are then presented in graphical

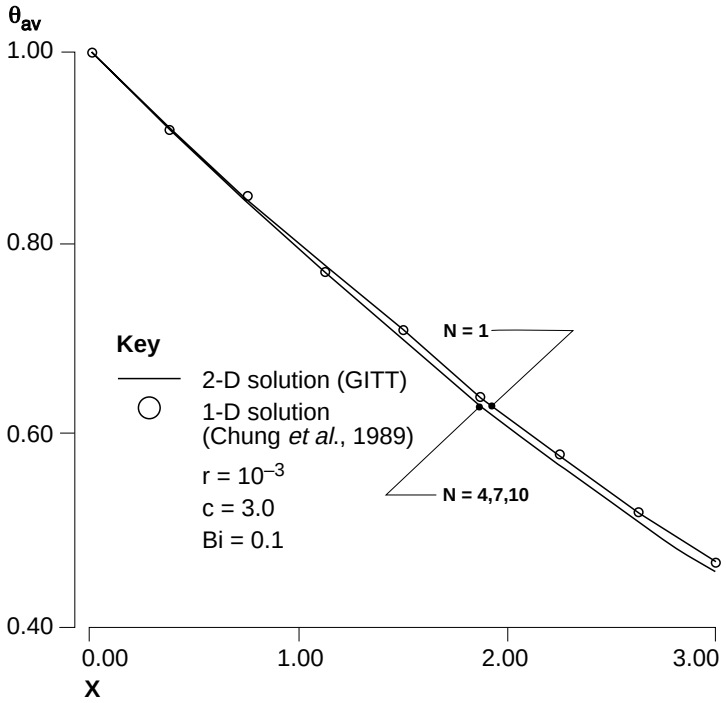

Figure 2.
Convergence of the integral transform solution for a trapezoidal longitudinal fin

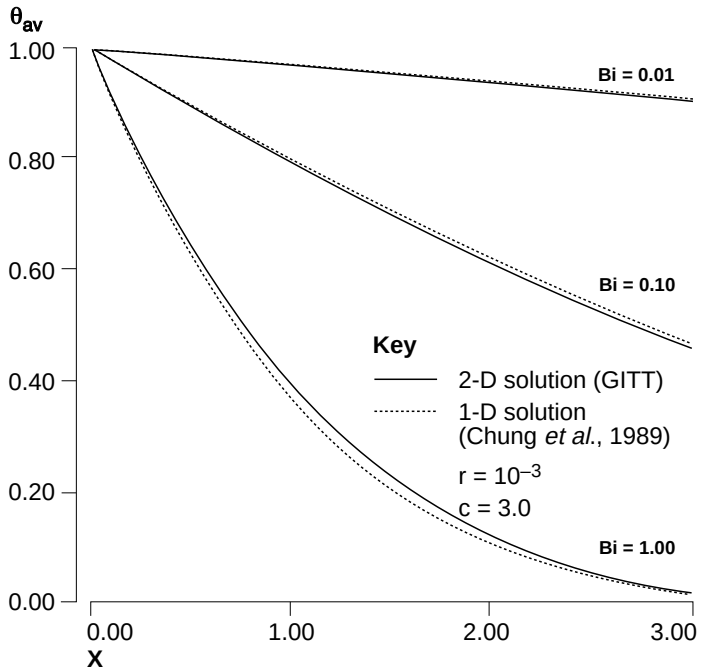

Figure 3.
Comparison of the two-dimensional and one-dimensional formulations for a trapezoidal longitudinal fin

form, to illustrate both the convergence rates and relative accuracy of the approximate solutions in the case of a parabolic profile. Figure 4 shows the convergence of the eigenfunction expansion for the average temperature along the fin length, together with the approximate one-dimensional solution with $C = 3.0$ and $Bi = 0.1$. The same trends observed for the trapezoidal profile are present in these comparisons for the parabolic geometry.

Also of interest is the evaluation of the dimensionless heat transfer rate at the fin base. Figure 5 presents a comparison of the heat transfer rates, as a function of aspect ratio C, and different values of $Bi = 0.1, 0.25$ and 0.5, for a trapezoidal fin with constant thermal conductivity ($b = 0$). The following definition was employed:

$$Q_0 = \frac{2}{\epsilon(0)} \int_0^{\epsilon(0)} \left. \frac{\partial \theta(X,Y)}{\partial X} \right|_{X=0} dY \tag{14a}$$

or,

$$Q_0 = 2 \left. \frac{d\theta_{av}(X)}{dX} \right|_{X=0} \tag{14b}$$

Besides the fully converged two-dimensional results, Figure 5 also shows the classical one-dimensional solution and the integral transform solution for one-single term ($N = 1$). Especially for the lower values of aspect ratio, the classical one-dimensional formulation deviates from the two-dimensional solution markedly, while the solution with $N = 1$ remains considerably accurate within this range of the Biot number.

Attention is now directed to the solution of the non-linear situation due to a temperature dependent thermal conductivity, for different values of the governing coefficient, b.

Table II illustrates the convergence rates of the eigenfunction expansion for a trapezoidal fin with variable thermal conductivity, with $Bi = 0.1$, $C = 10$ and $b = 0.01, 0.1, 1.0$. The average temperature results are in all cases fully converged to four digits with N as low as 4. Also shown are the results for the one-dimensional formulation, when the average value of the dimensionless thermal conductivity is adopted ($Kav = 1+b/2$), in the temperature range of the problem. Clearly, the results from the single-term eigenfunction expansion ($N = 1$), offer an excellent approximation of the two-dimensional formulation, with considerable accuracy improvement over the classical one-dimensional approach, and even more noticeably for increasing degree of non-linearity.

Figure 6 presents a set of reference results for the non-linear case, again for the trapezoidal fin with $Bi = 0.1$, $C = 10$ and $r = 10^{-3}$, and indicates the expected physical behaviour of the average temperature distributions as the non-linear effect is magnified from the constant thermal conductivity case, $b = 0$.

In conclusion, the integral transform approach has been demonstrated to be an attractive alternative to the purely numerical solution of non-linear heat

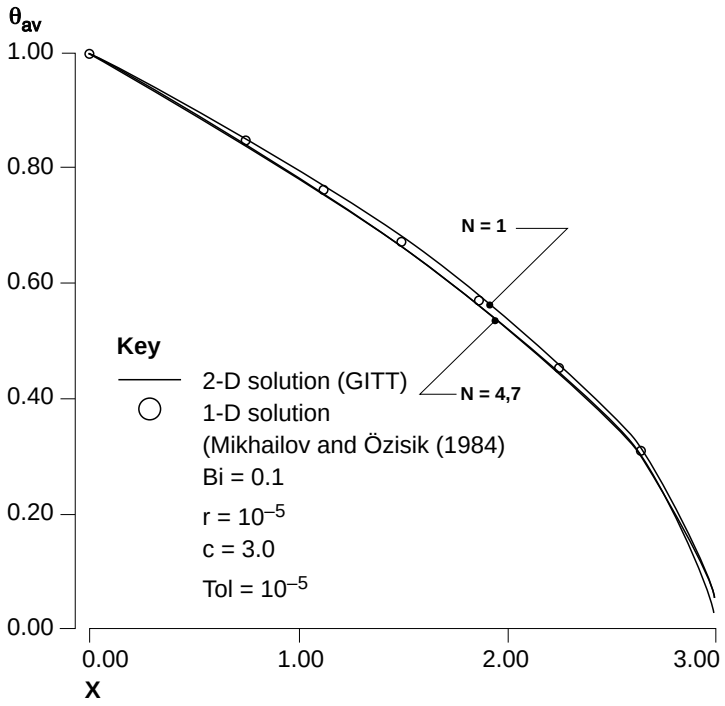

Figure 4. Convergence of the integral transform solution for a concave parabolic fin

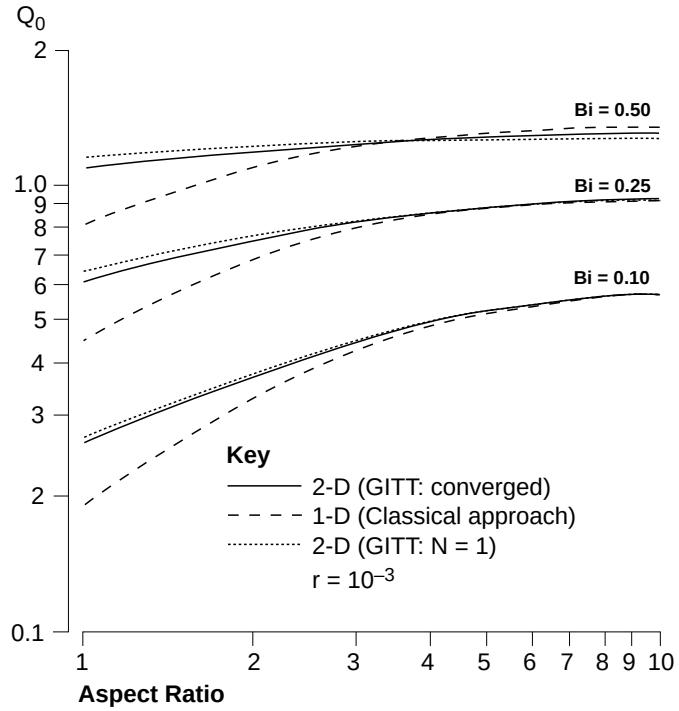

Figure 5. Comparison of the two-dimensional and one-dimensional formulations for a trapezoidal fin: dimensionless heat transfer rate

N	*X* = 0.000	*X* = 3.750	*X* = 6.250	*X* = 10.000
b = 1.00				
1 (% error)	1.000	0.4295(0.12)	0.1884(0.32)	0.0219(0.92)
4	1.000	0.4290	0.1879	0.0218
7	1.000	0.4290	0.1878	0.0217
10	1.000	0.4290	0.1878	0.0217
(*) (% error)	1.000	0.3830(–10.6)	0.1758(–6.39)	0.0322(48.4)
b = 0.10				
1(% error)	1.0000	0.3194(0.22)	0.1213(0.41)	0.0126(0.80)
4	1.0000	0.3187	0.1208	0.0125
7	1.0000	0.3187	0.1208	0.0125
10	1.0000	0.3187	0.1208	0.0125
(*) (% error)	1.0000	0.3099(–2.76)	0.1182(–2.15)	0.0130(4.00)
b = 0.01				
1 (% error)	1.0000	0.3066(0.23)	0.1150(0.44)	0.0119(0.85)
4	1.0000	0.3059	0.1145	0.0118
7	1.0000	0.3059	0.1145	0.0118
10	1.0000	0.3059	0.1145	0.0118
(*) (% error)	1.0000	0.3011(–1.57)	0.1120(–2.18)	0.0115(–2.54)

Note: (*) One-dimensional solution for *Kav*

Table II.
Convergence of the eigenfunction expansion for a trapezoidal fin with variable thermal conductivity (Bi = 0.10; C = 10.0; r = 10^{-3})

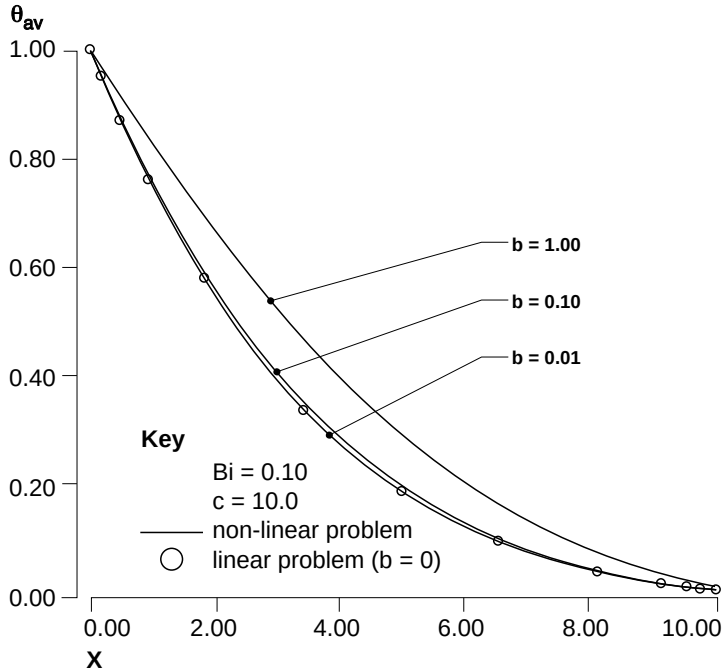

Figure 6.
Influence of the thermal conductivity coefficient on the average temperature profiles (trapezoidal fin)

conduction within extended surfaces of irregular geometry and variable thermal conductivity, providing rapidly converging representations of the full two-dimensional temperature field for different values of the governing parameters. This hybrid numerical-analytical method handles, in a straightforward manner, certain inherent difficulties associated with numerical methods, such as the singularities present at the tip of a triangular fin (Aparecido and Cotta, 1990b), and yields an automatic control of the global error in the final solution, just as in a purely analytical solution. In addition, the approximation based on one single term in the eigenfunction expansion was observed to be more consistent with the full two-dimensional solution than the classical one-dimensional fin solution, especially for higher Biot numbers and aspect ratios, offering a potential usefulness for more involved situations, particularly for non-linear situations, and keeping the same degree of mathematical complexity as for the classical one-dimensional formulation.

References

Aparecido, J.B. and Cotta, R.M. (1990a), "Improved one-dimensional fin solutions", *Heat Transfer Engineering,* Vol. 11 No. 1, pp. 49-59.

Aparecido, J.B. and Cotta, R.M. (1990b), "Laminar flow inside hexagonal ducts", *Computational Mechanics,* Vol. 6, pp. 93-100.

Aparecido, J.B. and Cotta, R.M. (1990c), "Analytical solution to parabolic multidimensional diffusion problems within irregular shaped domains", *International Conference on Advanced Computational Methods in Heat Transfer,* Vol. 1, Southampton, July, pp. 27-38.

Aparecido, J.B. and Cotta, R.M. (1992), "Laminar thermally developing flow inside right triangular ducts", *Appl. Scientific Research,* Vol. 49, pp. 355-68.

Aparecido, J.B., Cotta, R.M. and Ozisik, M.N. (1989), "Analytical solutions to two-dimensional diffusion-type problems in irregular geometries", *J. Franklin Inst.,* Vol. 326 No. 3, pp. 421-34.

Aziz, A. and Nguyen, H. (1992), "Two-dimensional effects in a triangular convecting fin", *J. Thermophysics & Heat Transfer,* Vol. 6 No.1, pp. 165-7.

Chung, B.T.F., Abdalla, M.H. and Liu, F. (1989), "Optimization of convective longitudinal fins of trapezoidal profile", *Chem. Eng. Comm,* Vol. 80, pp. 211-23.

Cotta, R.M. (1990), "Hybrid numerical-analytical approach to non-linear diffusion problems", *Num. Heat Transfer, Part B – Fundamentals,* Vol. 17, pp. 217-26.

Cotta, R.M. (1992), *Integral Transforms in Computational Heat and Fluid Flow,* CRC Press, Boca Raton, FL.

Cotta, R.M. and Serfaty, R. (1991), "Integral transform algorithm for parabolic diffusion problems with non-linear boundary and equation source terms", *7th International Conference on Numerical Methods for Thermal Problems, Part II,* Stanford, CA, July, pp. 916-26.

IMSL Library (1987), MATH/LIB, Houston, TX.

Irey, R.K. (1968), "Errors in the one-dimensional fin solution", *J. Heat Transfer,* Vol. 90, pp.175-6.

Kern, D.Q. and Kraus, A.D. (1972), *Extended Surface Heat Transfer,* McGraw-Hill, New York, NY.

Lau, W. and Tan, C.W. (1973), "Errors in the one-dimensional heat transfer analysis in straight and annular fins", *J. Heat Transfer,* Vol. 95, pp. 549-51.

Mikhailov, M.D. and Ozisik, M.N. (1984), *Unified Analysis and Solutions of Heat and Mass Diffusion,* John Wiley, New York, NY.

Ramos, R. (1993), "Two-dimensional non-linear analysis of longitudinal fins with variable profile via the generalized integral transform technique", MSc dissertation, Mechanical Engineering Department, COPPE/UFRJ, Rio de Janeiro.

Serfaty, R. and Cotta, R.M. (1990), "Integral transform solutions of diffusion problems with non-linear equation coefficients", *Int. Comm. Heat & Mass Transfer,* Vol. 17 No. 6, pp. 851-67.

Serfaty, R. and Cotta, R.M. (1992), "Hybrid analysis of transient non-linear convection-diffusion problems", *International Journal of Numerical Methods for Heat & Fluid Flow,* Vol. 2 No. 1, pp. 55-62.

Snider, A.D. and Kraus, A.D. (1983), "Recent developments in the analysis and design of extended surface", *J. Heat Transfer,* Vol. 105, pp. 302-6.